# Least Squares Regression of a Circle to Time Series Cartesian Data

W. Keith Adams, P. E.
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Derived is a least squares regression of a circle to uniform time series data of x and y coordinate pairs. A constant angular velocity is assumed. A solution using numerical methods is demonstrated.

The fit of a circle to pairs of x-y coordinates has an elegant closed-form solution described by Galer (1), and independently derived by others. The minimization of the variation of the radii is the basis of the derivation, and the result is known to be sensitive to noise. Specifically, the mean radius is usually attenuated by the presence of noise in either, or both, the x and y coordinates. An interactive animation by the author demonstrates this effect graphically.

Use of the time stamp allows for a formulation that is less sensitive to noise, but requires an iterative solution. The sum squared error between points uniformly spaced along an arc and the observed data is the objective function that is minimized. The form of the data need only be an arc, not a complete circle.

The sum of the squares objective function is:

$$\sum \varepsilon^2 = \sum_{i=1}^{n} \left[ \left(x_0 + r\cos(\omega t_i + \phi) - x_i\right)^2 + \left(y_0 + r\sin(\omega t_i + \phi) - y_i\right)^2 \right] \qquad (1)$$

where:

$x_0 =$ x coordinate of model circle center

$y_0 =$ y coordinate of fit circle center

$r =$ radius of fit circle

$\omega =$ angular velocity of fit circle

$\phi =$ phase angle of fit circle

$x_i =$ x coordinate of ith observation

$y_i =$ y coordinate of ith observation

$t_i =$ time at ith oservation

$\sum \varepsilon^2 =$ sum of the errors squared

$n =$ number of observations

The distance between each observation and its corresponding point along the arc is squared, and summed, creating the objective function.

The expanded objective function is:

$$\sum \varepsilon^2 = \sum_{i=1}^{n}\left[x_0^2 + y_0^2 + 2r(x_0 - x_i)\cos(\omega t_i + \phi) + 2r(y_0 - y_i)\sin(\omega t_i + \phi) - 2x_0 x_i - 2y_0 y_i + r^2 + x_i^2 + y_i^2\right] \tag{3}$$

The full derivation is in the Appendix. The partial derivatives result in these expressions for the model circle center and radius:

$$x_0 = \frac{\sum_{i=1}^{n}[x_i]}{n} - r\frac{\sum_{i=1}^{n}[\cos(\omega t_i + \phi)]}{n} \tag{7}$$

$$y_0 = \frac{\sum_{i=1}^{n}[y_i]}{n} - r\frac{\sum_{i=1}^{n}[\sin(\omega t_i + \phi)]}{n} \tag{10}$$

$$r = \frac{n\sum_{i=1}^{n} x_i C_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} C_i + n\sum_{i=1}^{n} y_i S_i - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} S_i}{n^2 - \sum_{i=1}^{n} C_i \sum_{i=1}^{n} C_i - \sum_{i=1}^{n} S_i \sum_{i=1}^{n} S_i} \tag{18}$$

Note that solution for the radius is not quite like the arithmetic mean in the Galer derivation.

The partial derivatives of $\phi$ and $\omega$ yield these two equations:

$$\sum_{i=1}^{n} x_i S_i - x_0 \sum_{i=1}^{n} S_i - \sum_{i=1}^{n} y_i C_i + y_0 \sum_{i=1}^{n} C_i = 0 \tag{22}$$

$$\sum_{i=1}^{n} x_i S_i t_i - x_0 \sum_{i=1}^{n} S_i t_i - \sum_{i=1}^{n} y_i C_i t_i + y_0 \sum_{i=1}^{n} C_i t_i = 0 \tag{26}$$

## Solution

1. Use Galer method to get initial value of center coordinates
2. Create vectors of angle and time from initial center coordinates and data
3. Unwrap angle-time vectors (ATAN2 returns angles between +/- 180 degrees)
4. Calculate initial values of angular velocity from linear regression of angle by time
5. Calculate initial values of phase angle from linear regression of angle by time
6. Use cascaded Newton's Method to find approximate zero values of (22) and (26)
7. Construct target data, including noise
8. Compare regression results with parameters used to construct target data

## **References**

Demonstration spreadsheet:  17_Circle_Fit_3_v09.xls

**Appendix A**

Derivation

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The fit of a circle to pairs of x-y coordinates has an elegant closed-form solution described by Galer (1), and independently derived by others (). The minimization of the variation of the radii is the basis of the derivation, and the result is known to be sensitive to noise (). Specifically, the mean radius is always attenuated by the presence of noise in either, or both, the x and y coordinates. An interactive animation by the author demonstrates this effect graphically.

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where:

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$y_0 =$ y coordinate of fit circle center

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$x_i =$ x coordinate of ith observation

$y_i =$ y coordinate of ith observation

$t_i =$ time at ith oservation

$\sum \varepsilon^2 =$ sum of the errors squared

$n =$ number of observations

The expanded variance equation is:

$$\sum \varepsilon^2 = \sum_{i=1}^{n} \left[ x_0^2 + 2x_0 r\cos(\omega t_i + \phi) - 2x_i r\cos(\omega t_i + \phi) - 2x_0 x_i + r^2\cos^2(\omega t_i + \phi) + x_i^2 \right] + \ldots$$

$$\ldots \sum_{i=1}^{n} \left[ y_0^2 + 2y_0 r\sin(\omega t_i + \phi) - 2y_i r\sin(\omega t_i + \phi) - 2y_0 y_i + r^2\sin^2(\omega t_i + \phi) + y_i^2 \right] \quad (2)$$

$$\sum \varepsilon^2 = \sum_{i=1}^{n} \left[ x_0^2 + y_0^2 + 2r(x_0 - x_i)\cos(\omega t_i + \phi) + 2r(y_0 - y_i)\sin(\omega t_i + \phi) - 2x_0 x_i - 2y_0 y_i + r^2 + x_i^2 + y_i^2 \right] \quad (3)$$

Take the partial derivative with respect to each of the unknowns and examine what substitutions are possible for the resulting system of equations.

First, partial derivatives with respect to the origin of the model circle:

$$\frac{\partial}{\partial x_0}\sum \varepsilon^2 = \sum_{i=1}^{n} \left[ 2(x_0 - x_i) + 2r\cos(\omega t_i + \phi) \right] = 0 \quad (4)$$

$$\sum_{i=1}^{n} \left[ (x_0 - x_i) + r\cos(\omega t_i + \phi) \right] = 0 \quad (5)$$

$$nx_0 = \sum_{i=1}^{n} [x_i] - r\sum_{i=1}^{n} [\cos(\omega t_i + \phi)] \quad (6)$$

$$x_0 = \frac{\sum_{i=1}^{n} [x_i]}{n} - r\frac{\sum_{i=1}^{n} [\cos(\omega t_i + \phi)]}{n} \quad (7)$$

$$\frac{\partial}{\partial y_0}\sum \varepsilon^2 = \sum_{i=1}^{n} \left[ 2(y_0 - y_i) + 2ry_0\sin(\omega t_i + \phi) \right] = 0 \quad (8)$$

$$ny_0 = \sum_{i=1}^{n} [y_i] - r\sum_{i=1}^{n} [\sin(\omega t_i + \phi)] \quad (9)$$

$$y_0 = \frac{\sum_{i=1}^{n} [y_i]}{n} - r\frac{\sum_{i=1}^{n} [\sin(\omega t_i + \phi)]}{n} \quad (10)$$

Take the partial derivative with respect to the radius of the circle:

$$\frac{\partial}{\partial r}\sum \varepsilon^2 = \sum_{i=1}^{n} \left[ 2(x_0 - x_i)\cos(\omega t_i + \phi) + 2(y_0 - y_i)\sin(\omega t_i + \phi) + 2r \right] = 0 \quad (11)$$

$$\sum_{i=1}^{n} \left[ (x_0 - x_i)\cos(\omega t_i + \phi) + (y_0 - y_i)\sin(\omega t_i + \phi) + r \right] = 0 \quad (12)$$

Let:

$C_i = \cos(\omega t_i + \phi)$ and $S_i = \sin(\omega t_i + \phi)$

$$\sum_{i=1}^{n}\left[(x_0 - x_i)C_i + (y_0 - y_i)S_i + r\right] = 0 \tag{13}$$

$$x_0\sum_{i=1}^{n}C_i - \sum_{i=1}^{n}x_iC_i + y_0\sum_{i=1}^{n}S_i - \sum_{i=1}^{n}y_iS_i + nr = 0 \tag{14}$$

Substitution of $x_0$ and $y_0$:

$$\left[\frac{\sum_{i=1}^{n}x_i}{n} - r\frac{\sum_{i=1}^{n}C_i}{n}\right]\sum_{i=1}^{n}C_i - \sum_{i=1}^{n}x_iC_i + \left[\frac{\sum_{i=1}^{n}y_i}{n} - r\frac{\sum_{i=1}^{n}S_i}{n}\right]\sum_{i=1}^{n}S_i - \sum_{i=1}^{n}y_iS_i + nr = 0 \tag{15}$$

$$\sum_{i=1}^{n}x_i\sum_{i=1}^{n}C_i - r\sum_{i=1}^{n}C_i\sum_{i=1}^{n}C_i - n\sum_{i=1}^{n}x_iC_i + \sum_{i=1}^{n}y_i\sum_{i=1}^{n}S_i - r\sum_{i=1}^{n}S_i\sum_{i=1}^{n}S_i - n\sum_{i=1}^{n}y_iS_i + n^2r = 0 \tag{16}$$

$$n^2r - r\sum_{i=1}^{n}C_i\sum_{i=1}^{n}C_i - r\sum_{i=1}^{n}S_i\sum_{i=1}^{n}S_i = n\sum_{i=1}^{n}x_iC_i - \sum_{i=1}^{n}x_i\sum_{i=1}^{n}C_i + n\sum_{i=1}^{n}y_iS_i - \sum_{i=1}^{n}y_i\sum_{i=1}^{n}S_i \tag{17}$$

$$r = \frac{n\sum_{i=1}^{n}x_iC_i - \sum_{i=1}^{n}x_i\sum_{i=1}^{n}C_i + n\sum_{i=1}^{n}y_iS_i - \sum_{i=1}^{n}y_i\sum_{i=1}^{n}S_i}{n^2 - \sum_{i=1}^{n}C_i\sum_{i=1}^{n}C_i - \sum_{i=1}^{n}S_i\sum_{i=1}^{n}S_i} \tag{18}$$

Note that solution for the radius is not quite like the arithmetic mean in the Galer derivation.

$$\sum\varepsilon^2 = \sum_{i=1}^{n}\left[x_0^2 + y_0^2 + 2r(x_0 - x_i)\cos(\omega t_i + \phi) + 2r(y_0 - y_i)\sin(\omega t_i + \phi) - 2x_0x_i - 2y_0y_i + r^2 + x_i^2 + y_i^2\right] \tag{3}$$

Take the partial derivative of (3) with respect to the phase angle of the circle:

$$\frac{\partial}{\partial\phi}\sum\varepsilon^2 = \sum_{i=1}^{n}\left[2r(x_i - x_0)\sin(\omega t_i + \phi) + 2r(y_0 - y_i)\cos(\omega t_i + \phi)\right] = 0 \tag{19}$$

$$\sum_{i=1}^{n}\left[(x_i - x_0)\sin(\omega t_i + \phi)\right] + \sum_{i=1}^{n}\left[(y_0 - y_i)\cos(\omega t_i + \phi)\right] = 0 \tag{20}$$

$$\sum_{i=1}^{n}(x_iS_i - x_0S_i) + \sum_{i=1}^{n}(y_0C_i - y_iC_i) = 0 \tag{21}$$

$$\sum_{i=1}^{n}x_iS_i - x_0\sum_{i=1}^{n}S_i - \sum_{i=1}^{n}y_iC_i + y_0\sum_{i=1}^{n}C_i = 0 \tag{22}$$

Since the solution will use Newton's Method, the partial derivative of (22) with respect to phi will be necessary. One could use an approximate derivative.

$$\frac{\partial}{\partial\phi}\left(\sum_{i=1}^{n}x_iS_i - x_0\sum_{i=1}^{n}S_i - \sum_{i=1}^{n}y_iC_i + y_0\sum_{i=1}^{n}C_i\right) = \sum_{i=1}^{n}x_iC_i - x_0\sum_{i=1}^{n}C_i - \left(\frac{\partial x_0}{\partial\phi}\right)\sum_{i=1}^{n}S_i + \sum_{i=1}^{n}y_iS_i - y_0\sum_{i=1}^{n}S_i + \left(\frac{\partial y_0}{\partial\phi}\right)\sum_{i=1}^{n}C_i \quad (23)$$

Take the partial derivative of (3) with respect to the angular velocity:

$$\frac{\partial}{\partial\omega}\sum\varepsilon^2 = \sum_{i=1}^{n}\left[2rt_i(x_i - x_0)\sin(\omega t_i + \phi) + 2rt_i(y_0 - y_i)\cos(\omega t_i + \phi)\right] = 0 \quad (24)$$

$$\sum_{i=1}^{n}\left[t_i(x_i - x_0)\sin(\omega t_i + \phi) + t_i(y_0 - y_i)\cos(\omega t_i + \phi)\right] = 0 \quad (25)$$

$$\sum_{i=1}^{n}x_iS_it_i - x_0\sum_{i=1}^{n}S_it_i - \sum_{i=1}^{n}y_iC_it_i + y_0\sum_{i=1}^{n}C_it_i = 0 \quad (26)$$

The partial derivative of (26) with respect to omega is:

$$\frac{\partial}{\partial\omega}\left(\sum_{i=1}^{n}x_iS_it_i - x_0\sum_{i=1}^{n}S_it_i - \sum_{i=1}^{n}y_iC_it_i + y_0\sum_{i=1}^{n}C_it_i\right) =$$
$$\sum_{i=1}^{n}x_iC_it_i^2 - x_0\sum_{i=1}^{n}C_it_i^2 - \frac{\partial x_0}{\partial\omega}\sum_{i=1}^{n}S_it_i + \sum_{i=1}^{n}y_iS_it_i^2 - y_0\sum_{i=1}^{n}S_it_i^2 + \frac{\partial y_0}{\partial\omega}\sum_{i=1}^{n}C_it_i \quad (27)$$

It is necessary to find the partial derivatives of r, $x_0$, and $y_0$ with respect to phi and omega, to find (23) and (27).

From (18):

$$\frac{\partial r}{\partial\phi} = \frac{-n\sum_{i=1}^{n}x_iS_i + \sum_{i=1}^{n}x_i\sum_{i=1}^{n}S_i + n\sum_{i=1}^{n}y_iC_i - \sum_{i=1}^{n}y_i\sum_{i=1}^{n}C_i}{n^2 - \sum_{i=1}^{n}C_i\sum_{i=1}^{n}C_i - \sum_{i=1}^{n}S_i\sum_{i=1}^{n}S_i} \quad (28)$$

$$\frac{\partial r}{\partial\omega} = \frac{-n\sum_{i=1}^{n}t_ix_iS_i + \sum_{i=1}^{n}x_i\sum_{i=1}^{n}t_iS_i + n\sum_{i=1}^{n}t_iy_iC_i - \sum_{i=1}^{n}y_i\sum_{i=1}^{n}t_iC_i}{n^2 - \sum_{i=1}^{n}C_i\sum_{i=1}^{n}C_i - \sum_{i=1}^{n}S_i\sum_{i=1}^{n}S_i} -$$
$$\frac{\left(2\sum_{i=1}^{n}C_i\sum_{i=1}^{n}t_iS_i - 2\sum_{i=1}^{n}t_iC_i\sum_{i=1}^{n}S_i\right)\left(n\sum_{i=1}^{n}y_iS_i - \sum_{i=1}^{n}y_i\sum_{i=1}^{n}S_i + n\sum_{i=1}^{n}x_iC_i - \sum_{i=1}^{n}x_i\sum_{i=1}^{n}C_i\right)}{\left(n^2 - \sum_{i=1}^{n}C_i\sum_{i=1}^{n}C_i - \sum_{i=1}^{n}S_i\sum_{i=1}^{n}S_i\right)^2} \quad (29)$$

From (7):

$$x_0 = \frac{\sum_{i=1}^{n}[x_i]}{n} - r\frac{\sum_{i=1}^{n}[\cos(\omega t_i + \phi)]}{n} \tag{7}$$

$$\frac{\partial x_0}{\partial \phi} = r\frac{\sum_{i=1}^{n} S_i}{n} - \frac{\partial r}{\partial \phi}\frac{\sum_{i=1}^{n} C_i}{n} \tag{30}$$

$$\frac{\partial x_0}{\partial \omega} = r\frac{\sum_{i=1}^{n} t_i S_i}{n} - \frac{\partial r}{\partial \omega}\frac{\sum_{i=1}^{n} C_i}{n} \tag{31}$$

From (10):

$$y_0 = \frac{\sum_{i=1}^{n}[y_i]}{n} - r\frac{\sum_{i=1}^{n}[\sin(\omega t_i + \phi)]}{n} \tag{10}$$

$$\frac{\partial y_0}{\partial \phi} = -r\frac{\sum_{i=1}^{n} C_i}{n} - \frac{\partial r}{\partial \phi}\frac{\sum_{i=1}^{n} S_i}{n} \tag{32}$$

$$\frac{\partial y_0}{\partial \omega} = -r\frac{\sum_{i=1}^{n} t_i C_i}{n} - \frac{\partial r}{\partial \omega}\frac{\sum_{i=1}^{n} S_i}{n} \tag{33}$$